§6.2 Direct Calculation of Anomalies

Let ψ be a column containing <u>all</u> left-handed fermion fields:

$$\psi = \begin{pmatrix} u_{\alpha L} \\ d_{\alpha L} \\ \nu_{eL} \\ e_L \\ \vdots \end{pmatrix}$$

where $\alpha = 1,2,3$ denotes color indices

Then we can combine these with their anti-particles obtained from charge conjugation:

$$C\psi C^{-1} = \underbrace{\tfrac{1}{2}(1+\gamma_5)}_{\text{left-handed projection}} \beta \mathcal{C} \psi^*$$

where $\beta \equiv i\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathcal{C} \equiv \gamma_2 \beta = -i \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}$

Using $\frac{1}{2}(1+\gamma_5)\beta\mathcal{C}\psi^* = \frac{1}{2}[\beta\,\mathcal{C}(1-\gamma_5)\psi]^*$ we can combine left-handed particles and anti-particles to

$$\chi \equiv \begin{bmatrix} \frac{1}{2}(1+\gamma_5)\psi \\ \frac{1}{2}[\beta\mathcal{C}(1-\gamma_5)\psi]^* \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1+\gamma_5)\psi \\ \frac{1}{2}(1+\gamma_5)\beta\mathcal{C}\psi^* \end{bmatrix}$$

$\rightarrow$ all components of χ belong to the $(\frac{1}{2},0)$ representation of Lorentz group

Under gauge trfs. we have:

$$\delta\psi = i\theta_\alpha\left[\tfrac{1}{2}(1+\gamma_5)t_\alpha^L + \tfrac{1}{2}(1-\gamma_5)t_\alpha^R\right]\psi$$

$$\rightarrow \quad \delta\chi = i\varepsilon_\alpha T_\alpha \chi,$$

where

$$T_\alpha = \begin{bmatrix} t_\alpha^L & 0 \\ 0 & -t_\alpha^{R*} \end{bmatrix} = \begin{bmatrix} t_\alpha^L & 0 \\ 0 & -(t_\alpha^R)^T \end{bmatrix}$$

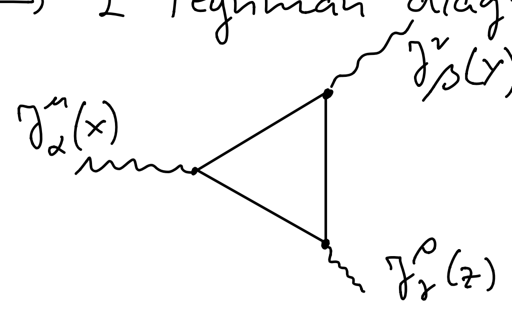
T_α will be any Hermitian representation of the gauge algebra (not necessarily block-diagonal)

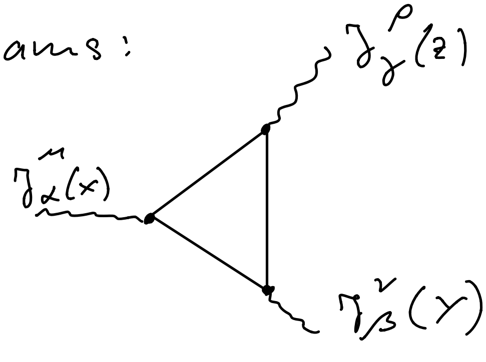
Consider the one-loop 3-point function:

$$\Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z) \equiv \langle T\{J^\mu_\alpha(x), J^\nu_\beta(y), J^\rho_\gamma(z)\}\rangle_{VAC},$$

where J^μ_α is the fermionic current, calculated in terms of free fields: $J^\mu_\alpha = -i\bar{\chi}T_\alpha\gamma^\mu\chi$

$\rightarrow$ 2 Feynman diagrams:





giving

$$\Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(x, y, z) \tag{1}$$

$$= -i\,\mathrm{Tr}\left[S(x-y)T_\beta \gamma^\nu P_L S(y-z) T_\gamma \gamma^\rho P_L S(z-x) T_\alpha \gamma^\mu P_L\right]$$

$$-i\,\mathrm{Tr}\left[S(x-z)T_\gamma \gamma^\rho P_L S(z-y) T_\beta \gamma^\nu P_L S(y-x) T_\alpha \gamma^\mu P_L\right]$$

where P_L is projection operator on left-handed fermions : $P_L = \left(\frac{1+\gamma_5}{2}\right)$

and $S(x)$ is the propagator of a massless fermion field:

$$S(x) = \frac{-i}{(2\pi)^4}\int d^4p \left(\frac{-i\not{p}}{p^2 - i\varepsilon}\right) e^{ip\cdot x}$$

Equation (1) then becomes

$$\Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(x, y, z) = \frac{i}{(2\pi)^{12}}\int d^4k_1\, d^4k_2\, e^{-i(k_1+k_2)}\, e^{ik_1\cdot y} e^{ik_2\cdot z}$$

$$\times \int d^4p \left\{ \mathrm{tr}\left[\frac{\not{p} - \not{k}_1 + \not{a}}{(p-k_1+a)^2 - i\varepsilon}\gamma^\nu \frac{\not{p}+\not{a}}{(p+a)^2 - i\varepsilon}\gamma^\rho \frac{\not{p}+\not{k}_2+\not{a}}{(p+k_2+a)^2 - i\varepsilon}\gamma^\mu \frac{1+\gamma_5}{2}\right]\right.$$

$$\times \mathrm{tr}\left[T_\beta T_\gamma T_\alpha\right]$$

$$+\ \mathrm{tr}\left[\frac{\not{p} - \not{k}_2 + \not{b}}{(p-k_2+b)^2 - i\varepsilon}\gamma^\rho \frac{\not{p}+\not{b}}{(p+b)^2 - i\varepsilon}\gamma^\nu \frac{\not{p}+\not{k}_1+\not{b}}{(p+k_1+b)^2 - i\varepsilon}\gamma^\mu \frac{1+\gamma_5}{2}\right]$$

$$\left.\times \mathrm{tr}\left[T_\gamma T_\beta T_\alpha\right]\right\}, \tag{2}$$

where "tr" here denotes a trace over Dirac or group indices

a and b are arbitrary constants

Using the identity

$$\cancel{k}_1 + \cancel{k}_2 = (\cancel{p} + \cancel{k}_2 + \cancel{a}) - (\cancel{p} - \cancel{k}_1 + \cancel{a})$$
$$= (\cancel{p} + \cancel{k}_1 + \cancel{b}) - (\cancel{p} - \cancel{k}_2 + \cancel{b})$$

and taking the divergence of (2), we find

$$\frac{\partial}{\partial x^\mu} \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(x, y, z) = \frac{1}{(2\pi)^{12}} \int d^4k_1\, d^4k_2\, e^{-i(k_1+k_2)\cdot x} e^{ik_1\cdot y} e^{ik_2\cdot z}$$

$$\times \int d^4p \Big\{ \mathrm{tr}\left[T_\beta T_\gamma T_\alpha\right] \mathrm{tr}\left[\frac{\cancel{p} - \cancel{k}_1 + \cancel{a}}{(p-k_1+a)^2 - i\varepsilon} \gamma^\nu \frac{\cancel{p}+\cancel{a}}{(p+a)^2 - i\varepsilon} \gamma^\rho \frac{1+\gamma_5}{2}\right]$$

$$- \mathrm{tr}\left[T_\beta T_\gamma T_\alpha\right] \mathrm{tr}\left[\frac{\cancel{p}+\cancel{a}}{(p+a)^2 - i\varepsilon} \gamma^\rho \frac{\cancel{p}+\cancel{k}_2+\cancel{a}}{(p+k_2+a)^2 - i\varepsilon} \gamma^\nu \frac{1+\gamma_5}{2}\right]$$

$$+ \mathrm{tr}\left[T_\gamma T_\beta T_\alpha\right] \mathrm{tr}\left[\frac{\cancel{p} - \cancel{k}_2 + \cancel{b}}{(p-k_2+b)^2 - i\varepsilon} \gamma^\rho \frac{\cancel{p}+\cancel{b}}{(p+b)^2 - i\varepsilon} \gamma^\nu \frac{1+\gamma_5}{2}\right]$$

$$- \mathrm{tr}\left[T_\gamma T_\beta T_\alpha\right] \mathrm{tr}\left[\frac{\cancel{p}+\cancel{b}}{(p+b)^2 - i\varepsilon} \gamma^\nu \frac{\cancel{p}+\cancel{k}_1+\cancel{b}}{(p+k_1+b)^2 - i\varepsilon} \gamma^\rho \frac{1+\gamma_5}{2}\right]\Big\} \quad (3)$$

Writing $\mathrm{tr}\left[T_\beta T_\gamma T_\alpha\right] = \underbrace{D_{\alpha\beta\gamma}}_{\text{sym.}} + \frac{1}{2} i N \underbrace{C_{\alpha\beta\gamma}}_{\text{anti-sym.}}$

and $\mathrm{tr}\left[T_\gamma T_\beta T_\alpha\right] = D_{\alpha\beta\gamma} - \frac{i}{2} N C_{\alpha\beta\gamma}$,

where
$$D_{\alpha\beta\gamma} = \frac{1}{2}\mathrm{tr}\left[\{T_\alpha, T_\beta\} T_\gamma\right]$$
and $\quad \mathrm{tr}[T_\alpha T_\beta] = N\,\delta_{\alpha\beta}\,.$

The anti-sym. terms correspond to time derivatives leading to equal-time commutation relations:

$$\left[\frac{\partial}{\partial x^\mu}\Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z)\right]_{formal}$$
$$= -iC_{\alpha\beta\delta}\,\delta^4(x-y)\left\langle \gamma^\nu_\delta(y)\,\gamma^\rho_\gamma(z)\right\rangle_{VAC}$$
$$-i\,C_{\alpha\gamma\delta}\,\delta^4(x-z)\left\langle \gamma^\nu_\beta(y)\,\gamma^\rho_\delta(z)\right\rangle_{VAC}$$

The anomaly is contained in the sym. part: grouping 1st and 4th traces, we get

$$\left[\frac{\partial}{\partial x^\mu}\Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z)\right]_{anom}$$
$$= \frac{1}{(2\pi)^{12}} D_{\alpha\beta\gamma}\int d^4k_1\, d^4k_2\, e^{-i(k_1+k_2)\cdot x}\, e^{ik_1\cdot y}\, e^{ik_2\cdot z}$$
$$\times\left\{\mathrm{tr}\left[\gamma^\kappa\gamma^\nu\gamma^\lambda\gamma^\rho\,\frac{1+\gamma_5}{2}\right] I_{\kappa\lambda}(a-b-k_1, b, b+k_1)\right.$$
$$\left.+\ \mathrm{tr}\left[\gamma^\kappa\gamma^\rho\gamma^\lambda\gamma^\nu\,\frac{1+\gamma_5}{2}\right] I_{\kappa\lambda}(b-a-k_2, a, a+k_2)\right\}, \tag{4}$$

where

$$I_{\kappa\lambda}(k,c,d) \equiv \int d^4p \left[f_{\kappa\lambda}(p+k,c,d) - f_{\kappa\lambda}(p,c,d) \right], \quad (5)$$

$$f_{\kappa\lambda}(p,c,d) \equiv \frac{(p+c)_\kappa (p+d)_\lambda}{[(p+c)^2 - i\varepsilon][(p+d)^2 - i\varepsilon]} \; .$$

To evaluate these integrals, expand f in powers of k:

$$f_{\kappa\lambda}(p+k,c,d) = \sum_{n=0}^{\infty} \frac{1}{n!} k^{\mu_1} \cdots k^{\mu_n} \frac{\partial^n f_{\kappa\lambda}(p,c,d)}{\partial p^{\mu_1} \cdots \partial p^{\mu_n}}$$

$\longrightarrow$ zeroth-order term cancels in (5)

After Wick-rotation all integrals can be written as surface integrals over large 3-sphere with radius P:

$$\int d^4p \underbrace{\partial_\mu F^\mu}_{\text{nth-derivative of } f_{\kappa\lambda}} = \int_{S^3} \vec{n} \cdot \underbrace{\vec{F}}_{= k^\mu \underbrace{\left[\frac{\partial}{\partial p^\nu}\right]^{n-1} f}_{\sim P^{-2-(n-1)}} \left[k^\nu\right]^{n-1}}$$

$$\propto P^3 \cdot P^{-2-(n-1)}$$

$\longrightarrow$ only terms that contribute for $P \to \infty$ are those with $n=1$ and $n=2$:

$$I_{\kappa\lambda}(k,c,d) = k^\mu \int d^4p \, \frac{\partial f_{\kappa\lambda}(p,c,d)}{\partial p^\mu} + \frac{1}{2} k^\mu k^\nu \int d^4p \, \frac{\partial^2 f_{\kappa\lambda}(p,c,d)}{\partial p^\mu \partial p^\nu}$$

A straightforward calculation then gives:

$$I_{\kappa\lambda}(k,c,d) = \frac{1}{6} i\pi^2 \Big[2k_\lambda c_\kappa + 2k_\kappa d_\lambda - k_\lambda d_\kappa - k_\lambda c_\lambda$$
$$- \eta_{\kappa\lambda}\, k\cdot(k+c+d) \Big]$$

The terms arising from 1 in $\frac{1}{2}(1+\gamma_5)$ appear in the combination:

$$\Big(I_{\kappa\lambda}(a-b-k_1, b, b+k_1) + I_{\lambda\kappa}(a-b-k_1, b, b+k_1)$$
$$+ I_{\kappa\lambda}(b-a-k_2, a, a+k_2) + I_{\lambda\kappa}(b-a-k_2, a, a+k_2) \Big) \underbrace{tr[\gamma^\kappa \gamma^\nu \gamma^\lambda \gamma^\rho]}_{\text{sym in } \kappa \leftrightarrow \lambda}$$

$\longrightarrow$ vanishes for $a = -b$

Left with the term involving γ_5:

$$tr[\gamma^\kappa \gamma^\nu \gamma^\lambda \gamma^\rho \gamma_5] = -4i\, \varepsilon^{\kappa\nu\lambda\rho}$$

$$\longrightarrow \left[\frac{\partial}{\partial x^\lambda} \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z) \right]_{anom}$$
$$= \frac{2}{(2\pi)^{12}} D_{\alpha\beta\gamma} \int d^4k_1\, d^4k_2\, e^{-i(k_1+k_2)\cdot x}$$
$$\times e^{ik_1\cdot y} e^{ik_2\cdot z}\, \pi^2\, \varepsilon^{\kappa\nu\lambda\rho} a_\kappa (k_1+k_2)_\lambda .$$

Taking $a = k_1 - k_2$ eliminates the anomaly in $\left(\frac{\partial}{\partial y^\nu}\right) \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z)$ and $\left(\frac{\partial}{\partial z^\rho}\right) \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z)$

$\rightarrow$ are left with

$$\left[\frac{\partial}{\partial x^{\mu}} \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(x,y,z)\right]_{\text{anom}}$$

$$= \frac{1}{(2\pi)^{12}} D_{\alpha\beta\gamma} \int d^4k_1\, d^4k_2\, e^{-i(k_1+k_2)\cdot x} e^{ik_1\cdot y} e^{ik_2\cdot z}$$

$$\times 4\pi^2 \varepsilon^{\kappa\nu\lambda\rho} k_{1\kappa} k_{2\lambda}$$

$$= -\frac{1}{4\pi^2} D_{\alpha\beta\gamma} \varepsilon^{\kappa\nu\lambda\rho} \frac{\partial \delta^4(y-x)}{\partial y^{\kappa}} \frac{\partial \delta^4(z-x)}{\partial z^{\lambda}}$$

next time have more to say on this...